# Runtime analysis of randomized local search (RLS) on the onemax function

- RLS $\longrightarrow$ like (1+1)-EA but flip exactly 1 bit in each iteration.

- We are interested in finding # function evaluations (on expectation) the until the algorithm reaches global optimum (all 1s).

- Let $n$ be the string length.

- Suppose current solution has $\underline{k\ 1s}$ and $\underline{n-k\ 0s}$.

- After doing one iteration, the probability that the child improves the parent's fitness equals the probability that we flip a 0 into a 1

$$= \frac{n-k}{n}$$

- Expected number of iterations to obtain an improvement $= \dfrac{1}{\text{Prob. improvement}} = \dfrac{n}{n-k}$

  $\hookrightarrow$ (Geometric Distribution)

- Worst case runtime occurs if the initial string is the string of all 0s.

- Let $p_k \equiv$ prob. that a solution with $k$ 1s improves after one iteration (i.e., improves to a solution with $k+1$ 1s)

- $$E[T] \leq \sum_{k=0}^{n-1} \frac{1}{p_k}$$

$$= \sum_{k=0}^{n-1} \frac{n}{n-k}$$

$$= n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1}\right)$$

$$= n \underbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)}_{= H_n \rightsquigarrow \text{Harmonic number}}$$

$$\ln(n) \leq H_n \leq \ln(n) + 1$$

$$\leq n\left(\ln(n) + 1\right)$$

$$= \boxed{\Theta(n \log n)}$$

# Similiar analysis for (1+1)-EA

- Assume $p_m = \frac{1}{n}$ (flip 1 bit on average)

- Assume current solution has $\underline{k}$ $\underline{1s}$ and $\underline{n-k}$ 0s

- Let $p_k$ = prob. of improvement after a single iteration.

- We obtain an improvement if we flip more 0s than 1s.

- Let's compute the probability of flipping exactly $i$ zeros and $j$ ones in one iteration.

$$\underbrace{\underbrace{\binom{n-k}{i}}_{\substack{\text{\# ways of}\\ \text{picking } i \text{ zeros}\\ \text{out of } n-k}} \underbrace{\left(\frac{1}{n}\right)^{i}}_{\substack{\text{prob. of}\\ \text{flipping}\\ i \text{ of them}}} \underbrace{\left(1-\frac{1}{n}\right)^{n-k-i}}_{\substack{\text{prob. of}\\ \text{not flipping}\\ \text{remaining}\\ n-k-i}}}_{\substack{\text{prob. flipping exactly}\\ i \text{ zeros}}} \underbrace{\binom{k}{j}\left(\frac{1}{n}\right)^{j}\left(1-\frac{1}{n}\right)^{k-j}}_{\substack{\text{prob. flipping exactly}\\ j \text{ ones}}}$$

- There's many ways of obtaining an improvement, but let's just consider ~~one~~ of them where we flip just one zero and don't flip any ~~ones~~ of the ones. (i.e., $i = 1, j = 0$)

- Previous formula becomes:

$$P_k \geq \binom{n-k}{1}\left(\frac{1}{n}\right)^{1}\left(1-\frac{1}{n}\right)^{n-k-1} \cancel{\binom{k}{0}\left(\frac{1}{n}\right)^{0}}\left(1-\frac{1}{n}\right)^{k}$$

$$= (n-k)\,\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$$

$$\geq (n-k)\,\frac{1}{n}\,\frac{1}{e}$$

$$= \frac{n-k}{ne}$$

$$\boxed{\left(1-\frac{1}{n}\right)^{k} \geq \frac{1}{e} \quad \text{for all } k \leq n-1}$$

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$$E[T] \leq \sum_{k=0}^{n-1} \frac{1}{p_k}$$

$$\leq \sum_{k=0}^{n-1} \frac{ne}{n-k}$$

$$= ne \sum_{k=0}^{n-1} \frac{1}{n-k}$$

$$= ne \left( \frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{1} \right)$$

$$= ne H_n$$

$$\leq ne \left( \ln(n) + 1 \right)$$

$$= \boxed{\Theta(n \log(n))}$$